Journal of Statistical Physics, Vol. 21, No. 5, 1979

Long-Time Behavior of the Lorentz Electron Gas in a Constant, Uniform Electric Field

J. Piasecki[1] **and E. Wajnryb**[1]

Received April 17, 1979

The long-time behavior of the Lorentz electron gas is studied in the presence of a uniform external field. A discussion of the rigorous solution of the one-dimensional Boltzmann equation is followed by the derivation of the asymptotic form of the velocity distribution in an arbitrary number of dimensions. The system is shown to absorb energy from the field without bounds, which excludes the usually assumed steady state with finite thermal energy density.

KEY WORDS: Lorentz gas; Boltzmann equation; distribution function.

1. INTRODUCTION

In 1905 H. A. Lorentz published an extensive study of the motion of electrons in metals based on a simple kinetic model.[1] Assuming that (i) electron–electron encounters may be neglected, (ii) the interaction between the electrons and the atoms in the metal can be approximated by collisions between hard spheres, and (iii) the atoms can be looked upon as immobile scattering centers, their masses being sufficiently big compared to the electron masses, he arrived at the corresponding Boltzmann equation for the electron distribution function

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right)f(\mathbf{r}, \mathbf{v}, t) = \pi n_A R^2|\mathbf{v}|(\hat{P} - 1)f(\mathbf{r}, \mathbf{v}, t) \qquad (1)$$

Here $\mathbf{r}$, $\mathbf{v}$, and t denote the position, velocity, and time, respectively, $\mathbf{E}$ is the acceleration due to the electric field, n_A stands for the number density of the scattering atoms, and $R = r_A + r_e$ is the sum of the atomic and electronic

Supported by the Polish Ministry of Higher Education, Science and Technology, Project MR.I.7.

[1] Institute of Theoretical Physics, Warsaw University, Warsaw, Poland.

0022-4715/79/1100-0549$03.00/0

radii. The collision term contains a projection operator $\hat{P} = \hat{P}^2$ which averages the distribution $f(\mathbf{r}, \mathbf{v}, t)$ over all directions in the velocity space

$$\hat{P}f(\mathbf{r}, \mathbf{v}, t) = \frac{1}{4\pi} \int d\Omega_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t) \tag{2}$$

The mean free path of the electrons is given by

$$\lambda = (\pi n_A R^2)^{-1} \tag{3}$$

Lorentz proposed a method for constructing an approximate stationary solution of Eq. (1), assuming it to have the form of the sum of a local Maxwell–Boltzmann distribution and an appropriate small correction. He argued that this correction would always remain small, provided the spatial temperature and density gradients, and also the electric field, were sufficiently weak. Most of the results known at his time in the theory of the electrical conduction in metals could then be derived, at least qualitatively, from this general standpoint. In particular he could reproduce the predictions of Drude's theory.

We shall show in this paper that the correctness of Lorentz's conclusions depends highly on the time scale involved, for, for physically relevant initial conditions, solutions of Eq. (1) tend eventually to a well-defined asymptotic distribution, completely different from the Maxwell–Boltzmann one. Of course, the classical Lorentz model is not used in the theory of metals any more. However, it still finds a number of applications, and, due to its simplicity, serves for illustrating the methods of the modern kinetic theory and testing various perturbation schemes.[2,3] From this point of view the knowledge of the long-time behavior of the distribution function satisfying Eq. (1) is certainly of interest. One can test, e.g., the so called two-term approximation used in solving the realistic Boltzmann equation for the electron motion.[4]

The electric field will be supposed here to be constant and uniform all over the system. To begin with, we present in Section 2 the analysis of the rigorous solution of a one-dimensional version of Eq. (1). We infer from it a method for determining the asymptotic form of the velocity distribution by an appropriate series expansion (Section 3), which is then applied to the three-dimensional case in Section 4. The interpretation of the final results and the possibility of applying the methods of this paper to more general problems is discussed in Section 5.

2. ONE-DIMENSIONAL CASE: THE EXACT SOLUTION

In one dimension, only two directions of velocity are possible. The action of the projector $\hat{P}$ thus reduces to replacing the complete distribution by its symmetric part

$$\hat{P}f(x, v, t) = f^s(x, v, t) = \tfrac{1}{2}[f(x, v, t) + f(x, -v, t)] \tag{4}$$

and Eq. (1) takes the form

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + E\frac{\partial}{\partial v}\right)f(x, v, t) = -\frac{|v|}{\lambda}f^{\mathrm{a}}(x, v, t) \tag{5}$$

where f^{a} denotes the antisymmetric part of f

$$f^{\mathrm{a}} = (1 - \hat{P})f \tag{6}$$

Due to the energy conservation the solution of the initial value problem for Eq. (5) can be reduced to that corresponding to the spatially homogeneous case. This follows from the remark that if $f_{v_0}(v, t)$ satisfies Eq. (5) with the initial condition

$$f_{v_0}(v, 0) = \delta(v - v_0) \tag{7}$$

then the distribution

$$f_{x_0 v_0}(x, v, t) = f_{v_0}(v, t)\,\delta\{(\tfrac{1}{2}v^2 - Ex - \tfrac{1}{2}v_0{}^2 + Ex_0)/E\} \tag{8}$$

is another solution of Eq. (5) corresponding to the initial condition

$$f_{x_0 v_0}(x, v, 0) = \delta(x - x_0)\,\delta(v - v_0) \tag{9}$$

Thus the conservation of the one-particle energy $\frac{1}{2}v^2 - Ex$, which appears in the argument of the Dirac δ in Eq. (8), determines here the spatial distribution once the velocity distribution is given. Keeping this in mind, we shall study from now on the homogeneous equation

$$\left(\frac{\partial}{\partial t} + E\frac{\partial}{\partial v}\right)f + \frac{|v|}{\lambda}f^{\mathrm{a}} = 0 \tag{10}$$

For the sake of simplicity the initial condition

$$f(v, 0) = \delta(v) \tag{11}$$

will be considered here [one can also find the rigorous solution of a more general problem (7)]. Equation (10) is equivalent to a system of two coupled equations for the symmetric and antisymmetric parts of f:

$$\left(\frac{\partial}{\partial t} + \frac{|v|}{\lambda}\right)f^{\mathrm{a}} + E\frac{\partial}{\partial v}f^{\mathrm{s}} = 0 \tag{12a}$$

$$\frac{\partial}{\partial t}f^{\mathrm{s}} + E\frac{\partial}{\partial v}f^{\mathrm{a}} = 0 \tag{12b}$$

where, in accordance with Eq. (11), we put

$$f^{\mathrm{s}}(v, 0) = \delta(v), \qquad f^{\mathrm{a}}(v, 0) = 0 \tag{13}$$

Equations (12a) and (12b) imply the following second-order equation for f^{a}:

$$\left(\frac{\partial^2}{\partial t^2} - E^2\frac{\partial^2}{\partial v^2} + \frac{|v|}{\lambda}\frac{\partial}{\partial t}\right)f^{\mathrm{a}} = 0 \tag{14}$$

Applying to it the Laplace transformation and taking into account the initial condition (13), we get

$$\left[\frac{\partial^2}{\partial v^2} - \frac{z}{\lambda E^2}(|v| + \lambda z)\right]\psi(v, z) = \frac{1}{E}\,\delta'(v) \tag{15}$$

where

$$\psi(v, z) = \int_0^\infty dt\, e^{-zt} f^{a}(v, t), \qquad \operatorname{Re} z > 0$$

and $\delta'(v)$ is the derivative of the δ distribution. Our aim is to find the antisymmetric solution of Eq. (15), integrable over velocity. To this end we consider the Airy equation

$$(d^2/ds^2 - s)\,\mathrm{Ai}(s) = 0 \tag{16}$$

which has two linearly independent solutions

$$\mathrm{Ai}_\pm(s) = s^{1/2} I_{\pm 1/3}(\tfrac{2}{3}s^{3/2}) \tag{17}$$

where $I_{1/3}$ and $I_{-1/3}$ are the modified Bessel functions with indices $\frac{1}{3}$ and $-\frac{1}{3}$. When $v > 0$, Eq. (15) reduces to Eq. (16) upon putting $s = (z/\lambda E^2)^{1/3}(v + \lambda z)$. It is thus clear that the antisymmetric solution will have the form

$$\begin{aligned}\psi(v, z) = \{2E\,\mathrm{Ai}[(z/\lambda E^2)^{1/3}\lambda z]\}^{-1}\{\theta(v)\,\mathrm{Ai}[(z/\lambda E^2)^{1/3}(v + \lambda z)] \\ - \theta(-v)\,\mathrm{Ai}[(z/\lambda E^2)^{1/3}(-v + \lambda z)]\}\end{aligned} \tag{18}$$

where θ is the Heaviside step function. The factor $\{2E\,\mathrm{Ai}[(z/\lambda E^2)^{1/3}\lambda z]\}^{-1}$ guarantees that $\psi(v, z)$ has the proper jump at $v = 0$, leading to the inhomogeneous term $E^{-1}\,\delta'(v)$ in Eq. (15). It turns out that the integrability condition determines (up to an unimportant constant factor) the choice of the solution of the Airy equation. The analysis of the behavior of the modified Bessel functions for $|z| \to \infty$ shows[5] that we have to put

$$\mathrm{Ai}(s) = s^{1/2} K_{1/3}(\tfrac{2}{3}s^{3/2}) \tag{19}$$

where

$$K_{1/3} = (\pi/3^{1/2})(I_{-1/3} - I_{1/3}) \tag{20}$$

is a modified Bessel function of the third kind (also called a Macdonald function). Equations (18)–(20) together yield the physically relevant solution of Eq. (15).

It is important to note that the function $\mathrm{Ai}(s)$ can be represented by a power series in the whole complex plane (see, e.g., the series representations of Bessel functions in Ref. 5). Moreover, at $s = 0$ we get $\mathrm{Ai}(0) = \frac{1}{2}3^{1/3}\Gamma(1/3) \neq 0$. This allows us to conclude that the function $\psi^{\mathrm{As}}(v, z)$ defined by

$$\begin{aligned}\psi^{\mathrm{As}}(v, z) = [2E\,\mathrm{Ai}(0)]^{-1}\{\theta(v)\,\mathrm{Ai}[(z/\lambda E^2)^{1/3}v] \\ - \theta(-v)\,\mathrm{Ai}[-(z/\lambda E^2)^{1/3}v]\}\end{aligned} \tag{21}$$

gives the proper asymptotic representation of the Laplace transform (18) of f^{a} for small values of z. It can thus be used for the determination of the asymptotic form of $f^{a}(v, t)$ for long times. In accordance with Eq. (19),

$$\mathrm{Ai}[(z/\lambda E^2)^{1/3}|v|] = |v|^{1/2}(z/\lambda E^2)^{1/6}K_{1/3}[\tfrac{2}{3}|v|^{3/2}(z/\lambda E^2)^{1/2}] \tag{22}$$

Using the remarkable result of the theory of integral transforms[6]

$$a^{-\nu/2}z^{\nu/2}K_\nu(2a^{1/2}z^{1/2}) = \frac{1}{2}\int_0^\infty dt\, e^{-zt}t^{-\nu-1}e^{-a/t}, \qquad \mathrm{Re}\, a > 0 \tag{23}$$

we can thus readily find the inverse Laplace transform of $\psi^{As}(v, z)$ by putting $\nu = \frac{1}{3}$ and $a = |v|^3/9\lambda E^2$. The resulting asymptotic form of f^{a} reads

$$f^{a}(v, t) = \frac{v}{2\Gamma(\frac{1}{3})Et(9\lambda E^2 t)^{1/3}} \exp \frac{-|v|^3}{9\lambda E^2 t} \tag{24}$$

Combining this result with Eq. (12), we find the corresponding asymptotic formula for the symmetric part of the velocity distribution

$$f^{s}(v, t) = \frac{3^{1/3}}{2\Gamma(\frac{1}{3})(\lambda E^2 t)^{1/3}} \exp \frac{-|v|^3}{9\lambda E^2 t} \tag{25}$$

It can be checked that the normalization condition

$$\int_{-\infty}^{+\infty} dv\, f^{s}(v, t) = 1 \tag{26}$$

is satisfied. In general, using the formula

$$\int_0^\infty dy \exp(-y^\mu) = \frac{1}{\mu}\Gamma\left(\frac{1}{\mu}\right), \qquad \mathrm{Re}\, \mu > 0 \tag{27}$$

we find

$$M_k(t) = \int_{-\infty}^{+\infty} dv\, |v|^k f(v, t) = (9\lambda E^2 t)^{k/3}\Gamma\left(\frac{k+1}{3}\right)/\Gamma\left(\frac{1}{3}\right) \sim t^{k/3} \tag{28}$$

Equation (28) implies that the kinetic energy of electrons tends to infinity as $t^{2/3}$. Clearly, they absorb it from the electric field. In Eq. (5) there is no mechanism for transmitting this energy to the atoms, and the system, being heated without bounds, cannot reach a steady-state regime with finite thermal energy density.

Equations (24) and (25) indicate that for long times it is natural to express the distribution $f(v, t)$ in terms of variables t and $u = vt^{-1/3}$. This remark will be of fundamental importance for the analysis carried out in the next two sections.

3. LONG-TIME EXPANSION OF THE VELOCITY DISTRIBUTION IN ONE DIMENSION

According to Eqs. (24) and (25), for long times the velocity distribution takes the form

$$f^{\mathrm{As}}(v, t) = t^{-1/3}[\phi_0(vt^{-1/3}) + t^{-2/3}\phi_1(vt^{-1/3})] \tag{29}$$

This suggests that one could study it in a more systematic way by using an expansion of the form

$$f(v, t) = t^{-1/3} \sum_{k=0}^{\infty} t^{-2k/3}\phi_k(vt^{-1/3}) \tag{30}$$

Let us denote by ϕ_k^{s} and ϕ_k^{a} the symmetric and antisymmetric parts of ϕ_k, respectively. Using variables t and $u = vt^{-1/3}$, and inserting expansion (30) into Eqs. (12a) and (12b), we get a hierarchy of equations for the functions ϕ_k^{s} and ϕ_k^{a} of the form

$$k = 0: \qquad \phi_0^{\mathrm{a}} = 0$$

$$E\frac{d}{du}\phi_0^{\mathrm{a}} = 0 \tag{31a}$$

$$k = 1: \qquad E\frac{d}{du}\phi_0^{\mathrm{s}} + \frac{|u|}{\lambda}\phi_1^{\mathrm{a}} = 0$$

$$\frac{d}{du}(u\phi_0^{\mathrm{s}}) - 3E\frac{d}{du}\phi_1^{\mathrm{a}} = 0 \tag{31b}$$

$$k = 2, 3,\ldots: \qquad u\frac{d}{du}\phi_{k-2}^{a} + (2k-3)\phi_{k-2}^{a} - 3E\frac{d}{du}\phi_{k-1}^{s} - 3\frac{|u|}{\lambda}\phi_k^{\mathrm{a}} = 0$$

$$u\frac{d}{du}\phi_{k-1}^{s} + (2k-1)\phi_{k-1}^{s} - 3E\frac{d}{du}\phi_k^{\mathrm{a}} = 0 \tag{31c}$$

They represent conditions for the vanishing of the coefficients of consecutive powers of variable $(t^{-2/3})$. The first pair of equations ($k = 0$) tells us that the function ϕ_0 is symmetric. The next pair ($k = 1$) yields a homogeneous system of equations for ϕ_0^{s} and ϕ_1^{a}. The second equation of this pair is equivalent to the relation

$$u\phi_0^{\mathrm{s}} - 3E\phi_1^{\mathrm{a}} = 0 \tag{32}$$

We thus find

$$3E^2\frac{d}{du}\phi_0^{\mathrm{s}} + \frac{u|u|}{\lambda}\phi_0^{\mathrm{s}} = 0 \tag{33}$$

which implies

$$\phi_0^{\mathrm{s}}(u) = C\exp(-|u|^3/9\lambda E^2) \tag{34}$$

Imposing the normalization condition (26), we determine the constant C, recovering formula (25). Then $\phi_1{}^a$ calculated from Eq. (32) reproduces the previous result (24).

Let us proceed further and examine the pair of equations corresponding to $k = 2$:

$$E\frac{d}{du}\phi_1{}^s + \frac{|u|}{\lambda}\phi_2{}^a = 0$$
$$u\frac{d}{du}\phi_1{}^s + 3\phi_1{}^s - 3E\frac{d}{du}\phi_2{}^a = 0 \tag{35}$$

The solution here reads

$$\phi_1{}^s(u) = C_1\frac{d}{du}\exp\frac{-|u|^3}{9\lambda E^2}\int_0^u dw \exp\frac{|w|^3}{9\lambda E^2}$$
$$\phi_2{}^a(u) = -\frac{E\lambda}{|u|}\frac{d}{du}\phi_1{}^s(u) \tag{36}$$

When $|u| \to \infty$, $\phi_1{}^s(u) \sim |u|^{-3}$. Hence, if we require that moments of ϕ^s exist [see Eq. (28)], we must put the constant C_1 equal to zero. In fact we could have assumed from the very beginning that $\phi^s_{2k+1} \equiv \phi^a_{2k} \equiv 0$, $k = 0, 1,...$, as these functions are not coupled by the hierarchy (31) to those with indices of opposite parity, i.e., ϕ^s_{2k} and ϕ^a_{2k+1}, $k = 0, 1,....$ We thus see that for a physically relevant class of distributions the first two leading terms in expansion (30) are entirely determined by $\phi_0{}^s$ and $\phi_1{}^a$, and do not depend on the initial condition. The same is then also true for the long-time asymptotic behavior of the moments of the velocity distribution. We conclude that the asymptotic state of the system in the region

$$t \to \infty, \qquad v \lesssim t^{1/3} \tag{37}$$

is well defined (i.e., independent of the state at $t = 0$), and corresponds the velocity distribution of the form

$$f^{As}(v, t) = \frac{3^{1/3}}{2\Gamma(\frac{1}{3})(\lambda E^2 t)^{1/3}}\left(1 + \frac{v}{3Et}\right)\exp\frac{-|v|^3}{9\lambda E^2 t} \tag{38}$$

We could in principle proceed further, solving consecutively pairs of equations in the hierarchy (31) with $k = 3, 4,....$ In this sense expansion (30) enables us to study systematically the long-time behavior of $f(v, t)$. What is even more interesting, it can be generalized in a straightforward way to three dimensions. This problem will be discussed in the next section.

4. LONG-TIME BEHAVIOR OF A THREE-DIMENSIONAL LORENTZ GAS

In a spatially homogeneous case Eq. (1) reduces to

$$\left(\frac{\partial}{\partial t} + \mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{v}, t) = \frac{|\mathbf{v}|}{\lambda}(\hat{P} - 1) f(\mathbf{v}, t) \tag{39}$$

If $f_{\mathbf{v}_0}$ satisfies Eq. (39) with the initial condition

$$f_{\mathbf{v}_0}(\mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0) \tag{40}$$

then the distribution

$$f_{(\mathbf{r}_0\cdot\mathbf{E})\mathbf{v}_0}(\mathbf{r}, \mathbf{v}, t) = f_{\mathbf{v}_0}(\mathbf{v}, t)\, \delta\{(\tfrac{1}{2}v^2 - \mathbf{E}\cdot\mathbf{r} - \tfrac{1}{2}v_0{}^2 + \mathbf{E}\cdot\mathbf{r}_0)/|\mathbf{E}|\} \tag{41}$$

represents the solution of Eq. (1) corresponding to the initial condition

$$f_{(\mathbf{r}_0\mathbf{E})\cdot\mathbf{v}_0}(\mathbf{r}, \mathbf{v}, 0) = \delta(\mathbf{v} - \mathbf{v}_0)\, \delta\{\mathbf{E}\cdot(\mathbf{r} - \mathbf{r}_0)/|\mathbf{E}|\} \tag{42}$$

This shows that knowing the evolution of the homogeneous solutions, one can also determine that of spatially inhomogeneous distributions, provided the inhomogeneity occurs only in the direction of the electric field. Energy conservation does not suffice to find the spatial distribution in the directions perpendicular to the vector $\mathbf{E}$ (there was no such problem in one dimension). Leaving this question open, we shall concentrate on establishing the long-time behavior of the homogeneous solutions. Proceeding as in the one-dimensional case, we define functions

$$f^{\mathrm{s}} = \hat{P}f, \qquad f^{\mathrm{a}} = (1 - \hat{P})f \tag{43}$$

and replace Eq. (39) by the system of equations

$$\begin{aligned} \frac{\partial}{\partial t} f^{\mathrm{s}} + \hat{P}\left(\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f^{\mathrm{a}} &= 0 \\ \frac{\partial}{\partial t} f^{\mathrm{a}} + (1 - \hat{P})\left(\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f^{\mathrm{a}} + \left(\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f^{\mathrm{s}} + \frac{|\mathbf{v}|}{\lambda} f^{\mathrm{a}} &= 0 \end{aligned} \tag{44}$$

The results of the previous sections indicate that in the asymptotic region (37) the distributions f^{s} and f^{a} take the form

$$t^{-1}\phi^{\mathrm{s}}(|\mathbf{v}|/t^{1/3}) \tag{45}$$

and

$$t^{-5/3}\phi^{\mathrm{a}}(\mathbf{v}/t^{1/3}) \tag{46}$$

respectively. The factor t^{-1} in (45) guarantees the time independence of the normalization of f:

$$\int d\mathbf{v}\, f(\mathbf{v}, t) = \int d\mathbf{u}\, \phi_0{}^{\mathrm{s}}(|\mathbf{u}|) = 1 \tag{47}$$

Inserting f^{s} and f^{a} in the forms (45) and (46) into Eq. (44), we find

$$\left(\frac{1}{3} u \frac{d}{du} + 1\right)\phi^{s} - \hat{P}\left(\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{u}}\right)\phi^{a} = 0$$

$$\mathbf{E}\cdot\frac{d}{d\mathbf{u}}\phi^{s} + \frac{u}{\lambda}\phi^{a} + \left[t^{-2/3}(1 - \hat{P})\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{u}}\phi^{a} - \frac{t^{-4/3}}{3}\left(5 + \mathbf{u}\cdot\frac{\partial}{\partial \mathbf{u}}\right)\phi^{a}\right] = 0 \tag{48}$$

where $\mathbf{u} = \mathbf{v}t^{-1/3}$, $u = |\mathbf{u}|$. The term in the brackets can be neglected in the long-time region (37). Hence

$$u^2\phi^{a} = -\lambda(\mathbf{E}\cdot\mathbf{u})\frac{d}{du}\phi^{s} \tag{49}$$

Using this relation and the formula

$$\hat{P}[(\mathbf{E}\cdot\mathbf{u})\mathbf{u}] = \tfrac{1}{3}u^2\mathbf{E} \tag{50}$$

we arrive at the following equation for ϕ^{s}:

$$\left[\lambda E^2 u \frac{d^2}{du^2} + (\lambda E^2 + u^3)\frac{d}{du} + 3u^2\right]\phi^{s}(u) = 0 \tag{51}$$

Its normalized solution reads

$$\phi^{s}(u) = (4\pi\lambda E^2)^{-1}\exp(-u^3/3\lambda E^2) \tag{52}$$

From Eqs. (45)–(49) we conclude that in three dimensions the long-time behavior of the velocity distribution satisfying Eq. (39) is given by

$$f^{As}(\mathbf{v}, t) = \frac{1}{4\pi\lambda E^2 t}\left(1 + \frac{\mathbf{E}\cdot\mathbf{v}}{E^2 t}\right)\exp\frac{-|\mathbf{v}|^3}{3\lambda E^2 t} \tag{53}$$

The dominant part of the distribution has spherical symmetry. Its moments follow the same power laws as in the one-dimensional case [see Eq. (28)]

$$M_k(t) \sim t^{k/3} \tag{54}$$

In particular, the second moment diverges as $t^{2/3}$, which reflects the fact that the energy absorbed by electrons from the field cannot be transferred to the atoms of the metal. This makes the model rather unphysical. One can also calculate the average power absorbed by an electron,

$$P(t) = \int d\mathbf{v}\, m(\mathbf{E}\cdot\mathbf{v})f^{As}(\mathbf{v}, t) = m\Gamma(\tfrac{5}{3})(\lambda^2 E^4/3t)^{1/3} \tag{55}$$

$P(t)$ tends to zero as $t^{-1/3}$, which shows the weakening of the influence of the field on the gas as the number of rapid particles increases.

5. DISCUSSION

Our results show that the Lorentz model of the electron gas in metals leads to an unrealistic behavior for long times. Even if the electric field is very weak, the velocity distribution will finally tend to its asymptotic form (53). At fixed mean free path λ the collision frequency tends to infinity (as $t^{1/3}$), and rapid collisions with spherical scattering centers make the dominant term in $f(\mathbf{v}, t)$ spherical, too. The distribution of velocities is asymptotically centered about $\mathbf{v} = 0$, and becomes flatter and flatter, diffusing gradually over the whole velocity space. No stationary regime with finite thermal energy density is possible.

This picture is completely different from that considered in Lorentz's paper,[1] showing the restricted validity of the arguments presented there. In fact Lorentz studied the distribution of the form

$$f(x, \mathbf{v}) = f^{\mathrm{MB}}(x, \mathbf{v}) + \delta f(x, \mathbf{v}) \tag{56}$$

where MB denotes Maxwell–Boltzmann, and

$$\delta f(x, \mathbf{v}) = -\lambda \left[\frac{v_x}{|\mathbf{v}|}\frac{\partial}{\partial x} + \frac{E}{|\mathbf{v}|}\frac{\partial}{\partial v_x}\right] f^{\mathrm{MB}}(x, \mathbf{v})$$

Considering a metallic wire, he assumed that the spatial inhomogeneity occurred only in the direction of the field (x direction). As shown in Section 4, this is precisely the situation that can be reduced to solving the homogeneous equation (39). And although for small spatial gradients and weak electric field the distribution (56) approximately satisfies Eq. (1), it will nevertheless be transformed (when one waits long enough) and lead to the asymptotic results of Section 4.

The solution of the form (56) has been also discussed by Wang Chang and Uhlenbeck,[3] who studied a much more general case with the external force field oscillating with a given frequency ω. Their results referring to the Lorentz gas (see Section 6 in Ref. 3) at $\omega = 0$ are again in contradiction with the asymptotic behavior established here, as they depend crucially on the assumption that the system stays close to the equilibrium Maxwell–Boltzmann distribution.

In order to solve Boltzmann's equation with realistic collision terms, one often assumes that the velocity distribution is almost spherical, and truncates the general expansion

$$f(\mathbf{r}, \mathbf{v}, t) = \sum_{k=0}^{\infty} f_k(\mathbf{r}, |\mathbf{v}|, t) P_k(\mathbf{E}\cdot\mathbf{v}/|\mathbf{E}|\,|\mathbf{v}|) \tag{57}$$

after the first two terms (the P_k are the Legendre polynomials).[4] Equation (53) shows that for the Lorentz gas this two-term approximation works perfectly well (at least for long times). On the other hand, the well-known relaxation time approximation does not make any sense in our case.

Before closing, let us indicate the possibility of generalizing our results. The methods used in this paper also can be applied at d dimensions ($d = 1, 2, \ldots$) for finding the long-time solutions of the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{v}, t) = \Lambda^{-1}|\mathbf{v}|^n(\hat{P} - 1) f(\mathbf{v}, t), \qquad n > -1 \tag{58}$$

The long-time expansion of the distribution f here takes the form

$$f(\mathbf{v}, t) = t^{-d/(n+2)} \sum_{k=0}^{\infty} t^{-(n+1)k/(n+2)} \phi_k(\mathbf{v} t^{-1/(n+2)}) \tag{59}$$

Following the analysis described in Sections 3 and 4, we find ($t \to \infty$, $|\mathbf{v}| \lesssim t^{1/(n+2)}$)

$$f^{\mathrm{As}}(\mathbf{v}, t) = C t^{-d/(n+2)} \left[1 + \frac{d(\mathbf{E}\cdot\mathbf{v})}{E^2(n+2)t}\right] \exp \frac{-d|\mathbf{v}|^{n+2}}{\Lambda E^2 t (n+2)^2} \tag{60}$$

The normalization condition yields

$$C = \frac{(n+2)\Gamma[d/2]}{2\pi^{d/2}\Gamma[d/(n+2)]} \left[\frac{d}{\Lambda E^2 (n+2)^2}\right]^{d/(n+2)} \tag{61}$$

It may be interesting to note that something peculiar must happen when $n \to -2$. Our analysis does not apply to this case. For any $n > -1$ the average energy diverges (when $t \to \infty$) as $t^{2/(n+2)}$.

ACKNOWLEDGMENT

The authors are very grateful to Dr. Y. Pomeau for many valuable comments.

REFERENCES

1. H. A. Lorentz, *Arch. Néerl.* **10**:336 (1905); see also H. A. Lorentz, *Collected Papers* (Martinus Nijhoff, The Hague, 1936), Vol. III.
2. E. H. Hauge, in *1974 International School of Statistical Mechanics, Sitges, Spain* (Springer-Verlag, Berlin, 1974), pp. 337–367.
3. C. S. Wang Chang and G. E. Uhlenbeck, in *Studies in Statistical Mechanics*, J. de Boer and G. E. Uhlenbeck, eds. (North-Holland, Amsterdam, 1970), Vol. V, Ch. V; for a recent study see J. L. Lebowitz and H. Spohn, *J. Stat. Phys.* **19**:633 (1978).
4. H. B. Milloy, *Austral. J. Phys.* **30**:73 (1977).
5. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed. (Academic Press, New York, 1965).
6. A. Erdélyi, ed., *Tables of Integral Transforms* (McGraw-Hill, New York, 1956), Vol. I.